A general Method of Summing Progressions *

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§1 In the last year I propounded a method to sum innumerable progressions, which method not only extends to series having algebraic sums but also exhibits sums depending on quadratures of curves, i.e. such which can not be summed algebraically. Then I used a synthetic method; for, having assumed general series I have been looking for series whose sums could be expressed by these formulas. And this way I obtained many general series whose sum I had been able to assign. Therefore, having propounded a progression which is to be summed it was necessary to compare it to those formulas and to decide whether it is contained in one of them. But I could have multiplied the number of the general series to infinity and therefore more often I encountered many series which, even though they are not comprehended in the given general ones, could nevertheless be summed by that method. Therefore, to find the sum of any arbitrary propounded series, if it is possible, of course, more easily and more quickly, I will explain a method here by which from the nature of the series the summatory term can be found. It extends very far; for, it not only teaches to find the sums of all the series whose sums have already been found in so many different ways but also those of infinitely many others by a similar and easy operation.

§2 If it would be as simple given the general term to find the summatory term as the general term from the summatory one, this would be an immense help

*Original title: "Methodus generalis summandi progressiones", first published in: Commentarii academiae scientiarum Petropolitanae 6, 1738, pp. 68-97, reprint in: Opera Omnia: Series 1, Volume 14, pp. 42 - 72, Eneström-Number E25, translated by: Alexander Aycock for the project „Euler-Kreis Mainz“.

in the summation of series. Certainly an equation between the summatory term and the general term can be given, but since it consists of infinitely many terms, from it we will not get a lot of help. But nevertheless, hence an extraordinary method to exhibit the sum of algebraic expressions results. Let the general term or the one whose exponent is n in the progression be t and let the summatory term or the sum of all terms from the first to t be $= s$; it will be

$$t = \frac{ds}{1dn} - \frac{dds}{1 \cdot 2dn^2} + \frac{d^3s}{1 \cdot 2 \cdot 3dn^3} - \frac{d^4s}{1 \cdot 2 \cdot 3 \cdot 4dn^4} + \text{etc.},$$

in which equation dn is put constant. But this equation can be transformed into this one

$$s = \int tdn + \alpha t + \frac{\beta dt}{dn} + \frac{\gamma d^2t}{dn^2} + \frac{\delta d^3t}{dn^3} + \text{etc.},$$

in which the coefficients α, β, γ etc. have the following values:

$$\begin{aligned}
\alpha &= \frac{1}{2}, \\
\beta &= \frac{\alpha}{2} - \frac{1}{6}, \\
\gamma &= \frac{\beta}{2} - \frac{\alpha}{6} + \frac{1}{24}, \\
\delta &= \frac{\gamma}{2} - \frac{\beta}{6} + \frac{\alpha}{24} - \frac{1}{120}, \\
\varepsilon &= \frac{\delta}{2} - \frac{\gamma}{6} + \frac{\beta}{24} - \frac{\alpha}{120} + \frac{1}{720} \\
& \text{etc.}
\end{aligned}$$

But it will be

$$s = \int tdn + \frac{t}{2} + \frac{dt}{12dn} - \frac{d^3t}{720dn^3} + \frac{d^5t}{30240dn^5} - \text{etc.}$$

Therefore, as often as t has a value of such kind that the series yielding s either terminates at some point or becomes summable, then s will be found from

t by means of this equation. That indeed happens, if t is a rational algebraic function of n and furthermore, if it is a fraction, if n just does not enter the denominator. For the sake of an example, let

$$t = n^2 + 2n;$$

it will be

$$dt = 2ndn + 2dn, \quad ddt = 2dn^2, \quad d^3t = 0 \quad \text{etc.}$$

Therefore, it will be

$$s = \int (n^2 + 2n)dn + \frac{n^2 + 2n}{2} + \frac{2n + 2}{12} = \frac{n^3}{3} + \frac{3n^2}{2} + \frac{7n}{6} = \frac{2n^3 + 9n^2 + 7n}{6}.$$

§**3** But the method I will present here works in such a way that by certain operations the propounded progression is reduced either to another simpler one, which can be summed, or to itself again; for, in each of both cases the sum of the propounded progression will become known. The operations, I use in these transformations, are either ordinary, as addition, subtraction etc., or taken from higher analysis, as differentiation or integration. They are certainly not only useful for such series whose summation is already known and can be assigned algebraically. Indeed, by them also sums depending on the quadratures of curves, i.e. sums of progressions having non-algebraic sums, are found. But all series this method can be accommodated to contain a geometric progression and have a form of this kind

$$\alpha x^a + \beta x^{a+b} + \gamma x^{a+2b} + \delta x^{a+3b} + \text{etc.}$$

This is not an obstruction for any progression to be contained in this form.

§**4** To start from the simplest cases, let this geometric progression be propounded

$$x^a + x^{a+b} + x^{a+2b} + x^{a+3b} + \cdots + x^{a+(n-1)b},$$

in which the last the term is the one whose index is n; and in the following it is always to be noted that the last term is the one whose index is n, that it is not necessary to write the index next to it; and hence I will always exhibit the sum up to the term with the index n. Put the sum of the propounded progression s; it will be

$$s = x^a + x^{a+b} + x^{a+2b} + \cdots + x^{a+(n-1)b};$$

then it will be

$$s - x^a = x^{a+b} + x^{a+2b} + \cdots + x^{a+(n-1)b};$$

add x^{a+nb} to both sides and divide by x^b; it will result

$$\frac{s - x^a + x^{a+nb}}{x^b} = x^a + x^{a+b} + \cdots + x^{a+(n-1)b} = s.$$

Therefore, we will have

$$s - x^a + x^{a+nb} = sx^b,$$

from which one finds

$$s = \frac{x^a - x^{a+nb}}{1 - x^b},$$

which is the sum of the propounded geometric progression. Therefore, this is an example in which the propounded progression is transformed into itself. If x was a fraction smaller than 1 and n is an infinitely large number, it will be $x^{a+nb} = 0$ and

$$s = \frac{x^a}{1 - x^b}$$

will give of the geometric progression

$$x^a + x^{a+b} + x^{a+2b} + \text{etc.}$$

continued to infinity. If $x = 1$, it is plain that $s = n$; this is indeed less apparent from the equation

$$s = \frac{x^a - x^{a+nb}}{1 - x^b},$$

since the numerator and the denominator vanish. To find the value in this case, put $x = 1 - \omega$ while ω denotes an infinitely small quantity; it will be

$$x^a = 1 - a\omega, \quad x^{a+nb} = 1 - (a+nb)\omega \quad \text{and} \quad x^b = 1 - b\omega.$$

And hence

$$s = \frac{nb\omega}{b\omega} = n.$$

It is also clear, if the general term of the series was $\alpha x^{a+(n-1)b}$, that the summatory term will be

$$\frac{\alpha x^a - \alpha x^{a+nb}}{1 - x^b}.$$

§5 Now let this progression be propounded

$$x^a + 2x^{a+b} + 3x^{a+2b} + \cdots + nx^{a+(n-1)b},$$

whose sum shall be put s. It will be

$$s - x^a = 2x^{a+b} + 3x^{a+2b} + \cdots + nx^{a+(n-1)b};$$

add the following term $(n+1)x^{a+nb}$ and divide by x^b; it will be

$$\frac{s - x^a + (n+1)x^{a+nb}}{x^b} = 2x^a + 3x^{a+b} + \cdots + (n+1)x^{a+(n-1)b}.$$

Subtract the first series, i.e. the propounded one, from this series; it will result

$$\frac{s - x^a + (n+1)x^{a+nb}}{x^b} - s = x^a + x^{a+b} + x^{a+2b} + \cdots + x^{a+(n-1)b} = \frac{x^a - x^{a+nb}}{1 - x^b}.$$

From this one finds

$$s = \frac{x^a - (n+1)x^{a+nb}}{1 - x^b} + \frac{x^{a+b} - x^{a+(n+1)b}}{(1 - x^b)^2} = \frac{x^a - (n+1)x^{a+nb} + nx^{a+(n+1)b}}{(1 - x^b)^2}$$

$$= \frac{x^a - x^{a+nb}}{(1 - x^b)^2} - \frac{nx^{a+nb}}{1 - x^b},$$

which is the summatory term corresponding to the general term $nx^{a+(n-1)b}$. If it was $x < 1$ and one puts $x = \infty$, the sum of the propounded series continued to infinity will result

$$= \frac{x^a}{(1-x^b)^2}.$$

But if $x = 1$, the sum of the following progression must result

$$1 + 2 + 3 + 4 + \cdots + n;$$

but here the same difficulty as before arises, i.e. that the numerator and denominator vanish; therefore, again I put $x = 1 - \omega$; it will be

$$1 - x^b = b\omega,$$

$$x^a = 1 - a\omega + \frac{a(a-1)\omega^2}{2},$$

$$x^{a+nb} = 1 - (a+nb)\omega + \frac{(a+nb)(a+nb-1)\omega^2}{2}$$

and

$$x^{a+(n+1)b} = 1 - (a+(n+1)b)\omega + \frac{(a+(n+1)b)(a+(n+1)b-1)\omega^2}{2}$$

and

$$s = \frac{(n^2b^2 + nb^2)\omega^2}{2b^2\omega^2} = \frac{n(n+1)}{2}.$$

Furthermore, if the general term is $\beta n x^{a+(n-1)b}$, the summatory term will be

$$= \frac{\beta x^a - \beta x^{a+nb}}{(1-x^b)^2} - \frac{\beta n x^{a+nb}}{1-x^b}.$$

§6 In like manner, one will find the summatory terms, if the general terms are

$$n^2 x^{a+(n-1)b}, \quad n^3 x^{a+(n-1)b} \quad \text{etc.};$$

for, the summation is always reduced to the summation of the series of lower degree. From this it is understood that this way in general one can find the summatory term corresponding to the general term

$$(\alpha + \beta n + \gamma n^2 + \text{etc.}) x^{a+(n-1)b}.$$

But I do not spend more time on doing this, since they are already well-known. I just mentioned these things, that the power of the method becomes clear through ordinary operations. Therefore, I proceed and will investigate which series can be summed by means of differentiation and integration. First certainly only the algebraic progressions just treated are summed and sums not differing from the ones already given are found; but nevertheless, the invention by means of these operations seems to be easier and shorter. For this reason I start from them again.

§7 Therefore, let this progression to be summed

$$x + 2x^2 + 3x^3 + 4x^4 + \cdots + nx^n;$$

put it $= s$, divide by x and multiply by dx; it will be

$$\frac{sdx}{x} = dx + 2xdx + 3x^2dx + \cdots + nx^{n-1}dx$$

and having taken integrals one has

$$\int \frac{sdx}{x} = x + x^2 + x^3 + \cdots + x^n = \frac{x - x^{n+1}}{1-x}.$$

Therefore, from the equation

$$\int \frac{sdx}{x} = \frac{x - x^{n+1}}{1-x},$$

after having differentiated it, one will find s. For, it will be

$$\frac{sdx}{x} = \frac{dx - (n+1)x^n dx + nx^{n+1}dx}{(1-x)^2},$$

whence it results

$$s = \frac{x - (n+1)x^{n+1} + nx^{n+1}}{(1-x)^2},$$

as before in § 5, if one writes 1 instead of a and b there.

From this it can be understood how the sum of the progression

$$ax^\alpha + (a+b)x^{\alpha+\beta} + (a+2b)x^{\alpha+2\beta} + \cdots + (a+(n-1)b)x^{\alpha+(n-1)\beta}$$

is to be found. For, put this sum in question s and multiply by $x^\pi dy$; it will be

$$x^\pi sdy = ax^{\alpha+\pi}dy + (a+b)x^{\alpha+\beta+\pi}dy + \cdots + (a+(n-1)b)x^{\alpha+(n-1)\beta+\pi}dy.$$

Now let

$$x^{\alpha+\pi} = y^a \quad \text{and} \quad x^{\alpha+\beta+\pi} = y^{a+b-1};$$

it will be

$$x^\beta = y^b \quad \text{and} \quad x = y^{b:\beta}.$$

And hence it will be

$$x^{\alpha+\pi} = y^{(\alpha+\pi)b:\beta} = y^{a-1}.$$

Therefore, it will be

$$\pi = \frac{\beta a - \alpha b - \beta}{b}$$

and

$$x^{\alpha+(n-1)\beta+\pi} = y^{a+(n-1)b-1}.$$

Having put these it will be

$$x^{\frac{\beta a-\alpha b-\beta}{b}} sdy = ay^{a-1}dy + (a+b)y^{a+b-1}dy + \cdots + (a+(n-1)b)y^{a+(n-1)b-1}dy$$

and having taken integrals

$$\int x^{\frac{\beta a-\alpha b-\beta}{b}} sdy = y^a + y^{a+b} + \cdots + y^{a+(n-1)b} = \frac{y^a - y^{a+nb}}{1-y^b}.$$

But since $y^b = x^\beta$, it will be

$$y = x^{\frac{\beta}{b}} \quad \text{and} \quad dy = \frac{\beta}{b} x^{\frac{\beta-b}{b}} dx$$

and having substituted these

$$\frac{\beta}{b} \int x^{\frac{\beta a-\alpha b-b}{b}} sdx = \frac{x^{\frac{a\beta}{b}} - x^{\frac{a\beta+nb\beta}{b}}}{1-x^\beta}.$$

The same equation can be found more easily without changing the variable x this way. Multiply the propounded progression by $px^{\pi}dx$; it will be

$$px^{\pi}sdx = pax^{\alpha+\pi}dx + \cdots + p(a+(n-1)b)x^{\alpha+(n-1)\beta+\pi}dx.$$

Determine p and π in such a way that

$$\alpha + (n-1)\beta + \pi = p(a+(n-1)b) - 1$$

or

$$\alpha + \pi + (n-1)\beta = ap + (n-1)bp - 1.$$

From this, since p and π cannot depend on n, two equations result

$$\beta = bp \quad \text{and} \quad \alpha + \pi = ap - 1,$$

whence

$$p = \frac{\beta}{b} \quad \text{and} \quad \pi = \frac{a\beta - \alpha b - b}{b}.$$

Having substituted these and taken integrals, as before, it will result

$$\frac{\beta}{b}\int x^{\frac{a\beta-\alpha b-b}{b}}sdx = x^{\frac{a\beta}{b}} + x^{\frac{a\beta+b\beta}{b}} + \cdots + x^{\frac{a\beta+(n-1)b\beta}{b}} = \frac{x^{\frac{a\beta}{b}} - x^{\frac{a\beta+nb\beta}{b}}}{1-x^{\beta}}.$$

§8 Let the term of order n of the propounded progression be

$$(an+b)(cn+e)x^{\alpha+(n-1)\beta};$$

but the summatory term of this progression s; it will be

$$s = (a+b)(c+e)x^{\alpha} + (2a+b)(2c+e)x^{\alpha+\beta} + \cdots + (an+b)(cn+e)x^{\alpha+(n-1)\beta};$$

multiply it by $px^{\pi}dx$; it will be

$$psx^{\pi}dx = p(a+b)(c+e)x^{\alpha+\pi}dx + \cdots + p(an+b)(cn+e)x^{\alpha+(n-1)\beta+\pi}dx.$$

Let

$$pcn + pe = \alpha + n\beta - \beta + \pi + 1;$$

it must be

$$p = \frac{\beta}{c} \quad \text{and} \quad \pi = \frac{\beta e + \beta c - \alpha c - c}{c}.$$

Therefore, having taken integrals it will be

$$\frac{\beta}{c}\int x^{\pi} s dx = (a+b)x^{\alpha+\pi+1} + \cdots + (an+b)x^{\alpha+(n-1)\beta+\pi+1}.$$

Multiply by $qx^{\rho}dx$ again; it will be

$$\frac{\beta}{c} qx^{\rho}dx \int x^{\pi} s dx = q(a+b)x^{\alpha+\pi+\rho+1}dx + \cdots + q(an+b)x^{\alpha+(n-1)\beta+\pi+\rho+1}dx$$

and let

$$anq + bq = \alpha + n\beta - \beta + \pi + \rho + 2;$$

hence it will be

$$q = \frac{\beta}{a} \quad \text{and} \quad \rho = \frac{\beta b - \alpha a + \beta a - \pi a - 2a}{a} = \frac{\beta bc - ac - \beta ae}{ac}.$$

And having taken integrals it will result

$$\frac{\beta^2}{ac}\int x^{\rho}dx \int x^{\pi} s dx = x^{\alpha+\pi+\rho+2} + \cdots + x^{\alpha+(n-1)\beta+\pi+\rho+2} = \frac{x^{\alpha+\pi+\rho+2} - x^{\alpha+n\beta+\pi+\rho+2}}{1 - x^{\beta}}$$

or this equation

$$\frac{\beta^2}{ac}\int x^{\frac{\beta bc - \beta ae - ac}{ac}}dx \int x^{\frac{\beta e + \beta c - \alpha c - c}{c}} s dx = \frac{x^{\frac{\beta(a+b)}{a}} - x^{\frac{\beta(a+b+na)}{a}}}{1 = x^{\beta}} = x^{\frac{\beta(a+b)}{a}} \frac{1 - x^{n\beta}}{1 - x^{\beta}}.$$

In like manner, the operation is to be done, if more than two factors appeared in the general term, from which it is clear at the same time that as many integral signs result as there are factors in the general term.

§**9** If the general term of the progression which is to be summed was

$$\frac{x^{\alpha+(n-1)\beta}}{an+b},$$

the operation differs from the first only in that regard that here one must do by differentiation what had to be done by taking integrals there. Therefore, let the summatory term in question be s; it will be

$$s = \frac{x^{\alpha}}{a+b} + \cdots + \frac{x^{\alpha+(n-1)\beta}}{an+b}$$

and

$$px^{\pi}s = \frac{px^{\alpha+\pi}}{a+b} + \cdots + \frac{px^{\alpha+(n-1)\beta+\pi}}{an+b}.$$

Take the differentials; it will result

$$px^{\pi}ds + p\pi x^{\pi-1}sdx = \frac{p(\alpha+\pi)x^{\alpha+\pi-1}dx}{a+b} + \cdots + \frac{p(\alpha+n\beta-\beta+\pi)x^{\alpha+(n-1)\beta+\pi-1}dx}{an+\beta}.$$

Let

$$p\alpha + pn\beta - p\beta + p\pi = an+b;$$

it will be

$$p = \frac{a}{\beta} \quad \text{and} \quad \pi = \beta - \alpha + \frac{b\beta}{a}.$$

Therefore,

$$\frac{ax^{\beta-\alpha+\frac{b\beta}{a}}ds + (a\beta - a\alpha + b\beta)x^{\beta-\alpha+\frac{b\beta}{a}-1}sdx}{\beta dx} = x^{\frac{a\beta+b\beta-1}{a}} + \cdots + x^{\frac{na\beta+b\beta-a}{a}}$$

$$= x^{\frac{a\beta+b\beta-a}{a}}\frac{1-x^{n\beta}}{1-x^{\beta}}$$

or

$$\frac{a}{\beta}x^{\frac{a\beta-a\alpha+b\beta}{a}}s = \int x^{\frac{a\beta+b\beta-a}{a}}dx\frac{1-x^{n\beta}}{1-x^{\beta}}$$

or

$$s = \frac{\beta}{a} x^{\frac{a\alpha - a\beta - b\beta}{a}} \int x^{\frac{a\beta + b\beta - a}{a}} dx \frac{1 - x^{n\beta}}{1 - x^{\beta}}.$$

In this formula the integral must be taken in such a way that it vanishes for $x = 0$. If the sum of the propounded series continued to infinity is desired, it will be $n = \infty$ and

$$s = \frac{\beta}{a} x^{\frac{a\alpha - a\beta - b\beta}{a}} \int \frac{x^{\frac{a\beta + b\beta - a}{a}} dx}{1 - x^{\beta}}.$$

If $x = 1$, in the expression of the sum s, since there are differentials in it, one can not set $x = 1$, but after the integration let $x = 1$. But nevertheless it does not matter which numbers are substituted for α and β; therefore, let $\alpha = \beta = 1$. It will be

$$s = \frac{1}{a+b} + \frac{1}{2a+b} + \cdots + \frac{1}{na+b} = \frac{1}{ax^{\frac{b}{a}}} \int x^{\frac{b}{a}} dx \frac{1 - x^n}{1 - x}.$$

And after the integration x must become $= 1$. As I had found in the dissertation on summations cited initially.

§10 Let a progression be propounded whose term of order n is

$$\frac{x^n}{(an+b)(cn+e)};$$

here I only assume x^{π} instead of $x^{\alpha + (n-1)\beta}$ so for the sake of brevity as since this power can easily transformed into that one. Let the summatory term be s; it will be

$$px^{\pi} s = \frac{px^{\pi+1}}{(a+b)(c+e)} + \cdots + \frac{px^{\pi+n}}{(an+b)(cn+e)}$$

and hence

$$\frac{d.(px^{\pi} s)}{dx} = \frac{p(\pi+1)x^{\pi}}{(a+b)(c+e)} + \cdots + \frac{p(\pi+n)x^{\pi+n-1}}{(an+b)(cn+e)}.$$

Let

$$p\pi + pn = an + b;$$

it will be

$$p = a \quad \text{and} \quad \pi = \frac{b}{a}.$$

Therefore, one has

$$\frac{ad\left(x^{\frac{b}{a}}s\right)}{dx} = \frac{x^{\frac{b}{a}}}{c+e} + \cdots + \frac{x^{\frac{b}{a}+n-1}}{cn+e}.$$

Multiply by px^{π} again; it will be

$$\frac{apx^{\pi}d\left(x^{\frac{b}{a}}s\right)}{dx} = \frac{px^{\frac{b}{a}+\pi}}{c+e} + \cdots + \frac{px^{\frac{b}{a}+n+\pi-1}}{cn+e}.$$

And hence it results

$$\frac{apd\left(x^{\pi}d\left(x^{\frac{b}{a}}s\right)\right)}{dx^2} = \frac{p\left(\frac{b}{a}+\pi\right)x^{\frac{b}{a}+\pi-1}}{c+e} + \cdots + \frac{p\left(\frac{b}{a}+n+\pi-1\right)x^{\frac{b}{a}+n+\pi-2}}{cn+e}.$$

Let

$$\frac{pb}{a} + pn + p\pi - p = cn + e;$$

it will be

$$p = c \quad \text{and} \quad \pi = 1 - \frac{b}{a} + \frac{e}{c}.$$

Having substituted these this equation will emerge

$$\frac{acd\left(x^{1-\frac{b}{a}+\frac{e}{c}}d\left(x^{\frac{b}{a}}s\right)\right)}{dx^2} = x^{\frac{e}{c}} + \cdots + x^{\frac{e}{c}+n-1} = x^{\frac{e}{c}}\frac{1-x^n}{1-x}.$$

Take the integrals again; it will be

$$\frac{acx^{1-\frac{b}{a}+\frac{e}{c}}d\left(x^{\frac{b}{a}}s\right)}{dx} = \int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}$$

and hence

$$s=\frac{1}{acx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}=\frac{x^{\frac{b}{a}-\frac{e}{c}}\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}-\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}}{(bc-ae)x^{\frac{b}{a}}}.$$

Here the case is to be noted in which $bc=ae$ and in which $s=\frac{0}{0}$. But according to the first form it will be

$$s=\frac{1}{acx^{\frac{b}{a}}}\int\frac{dx}{x}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x},$$

which is changed into this one

$$s=\frac{\log x\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}-\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}\log x}{acx^{\frac{b}{a}}}.$$

This case occurs, if the denominators $(an+b)(cn+e)$ are squares or certain multiples of these. If $x=1$, this substitution, as before, must happen just after the integration in the quantities having integral signs in front of them, but in the finite ones one can put $x=1$ immediately. Therefore, it will be

$$s=\frac{\int\left(x^{\frac{e}{c}}-x^{\frac{b}{a}}\right)dx\frac{1-x^n}{1-x}}{bc-ae}.$$

From this it is clear, if $x^{\frac{e}{c}}-x^{\frac{b}{a}}$ can be divided by $1-x$, that the sum of the progression is algebraic. But in the case in which $bc=ae$ it will be $\log x=0$, if $x=1$, of course. Therefore, it will be

$$s=-\frac{\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}\log x}{ac}.$$

§11 In like manner, it is understood, if n has three or more dimensions in the denominator, how the sum must be found, such that it not necessary to illustrate the operation in more examples. Let this progression be propounded, whose general term is

$$\frac{x^n}{(an+b)(cn+e)(fn+g)};$$

let its sum be s. This progression treated in the same way as in the preceding § after two differentiations will give (§ 9)

$$\frac{acd\left(x^{1-\frac{b}{a}-\frac{e}{c}}d\left(x^{\frac{b}{a}}s\right)\right)}{dx^2}=\frac{x^{\frac{e}{c}}}{f+g}+\cdots+\frac{x^{\frac{e}{c}+n-1}}{nf+g}=\frac{1}{f}x^{\frac{e}{c}-\frac{g}{f}-1}\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x};$$

take integrals; it will be

$$\frac{acfx^{1-\frac{b}{a}-\frac{e}{c}}d\left(x^{\frac{b}{a}}s\right)}{dx}=\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}$$

and again

$$acfx^{\frac{b}{a}}s=\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}$$

and hence

$$s=\frac{1}{acfx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}.$$

That not more integral signs are written after each other, this form can be transformed into the following

$$s=\frac{fx^{-\frac{g}{f}}\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}}{(bf-ag)(ef-cg)}+\frac{cx^{-\frac{e}{c}}\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}}{(bc-ae)(cg-ef)}+\frac{ax^{-\frac{b}{a}}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}}{(ae-bc)(ag-bf)}.$$

From this it is clear at the same time, if there were more factors in the general term, which form the sum will have. For, let the general term be

$$\frac{x^n}{(an+b)(cn+e)(fn+g)(hn+k)};$$

the summatory term will be

$$\begin{aligned}s&=\frac{1}{acfhx^{\frac{b}{a}}}\int x^{\frac{b}{a}-\frac{e}{c}-1}dx\int x^{\frac{e}{c}-\frac{g}{f}-1}dx\int x^{\frac{g}{f}-\frac{k}{h}-1}dx\int x^{\frac{k}{h}}dx\frac{1-x^n}{1-x}\\&=\frac{ax^{-\frac{b}{a}}\int x^{\frac{b}{a}}dx\frac{1-x^n}{1-x}}{(ae-bc)(ag-bf)(ak-bh)}+\frac{cx^{-\frac{e}{c}}\int x^{\frac{e}{c}}dx\frac{1-x^n}{1-x}}{(bc-ae)(cg-ef)(ck-eh)}\\&+\frac{fx^{-\frac{g}{f}}\int x^{\frac{g}{f}}dx\frac{1-x^n}{1-x}}{(bf-ag)(ef-cg)(fk-gh)}+\frac{hx^{-\frac{k}{h}}\int x^{\frac{k}{h}}dx\frac{1-x^n}{1-x}}{(bh-ak)(eh-ck)(gh-fk)}.\end{aligned}$$

If the sum is desired in the case $x = 1$, for the general term

$$\frac{1}{(an+b)(cn+e)(fn+g)}$$

the summatory term will be

$$s = \frac{\int dx \frac{1-x^n}{1-x}\left((aef-bcf)x^{\frac{f}{g}} + (bcf-acg)x^{\frac{e}{c}} + (acg-aef)x^{\frac{b}{a}}\right)}{(ae-bc)(ag-bf)(cg-ef)}.$$

Therefore, as often as the quantity multiplied by $dx\frac{1-x^n}{1-x}$ can be divided by $1-x$, the propounded progression will have an algebraic sum. This happens, if $\frac{b}{a} - \frac{e}{c}$ and $\frac{e}{c} - \frac{g}{f}$ are integer numbers. Furthermore, it is also to be noted that all progressions of this kind either are summable algebraically or depend on either real or imaginary logarithms and no other quadrature of this kind can be expressed by a progression of this kind.

§**12** But since it is difficult to accommodate these formulas to those in which the factors of the denominators are equal, it will be convenient to consider them here separately. Therefore, let the general term of the progression which is to be summed be

$$\frac{x^n}{(an+b)^3}$$

and the summatory term s; it will be

$$s = \frac{1}{a^3 x^{\frac{b}{a}}} \int \frac{dx}{x} \int \frac{dx}{x} \int x^{\frac{b}{a}} dx \frac{1-x^n}{1-x},$$

which follows from § 11, where $c = f = a$ and $e = g = b$; this form if transformed goes over into this one

$$s = \frac{\frac{1}{2}(\log x)^2 \int x^{\frac{b}{a}} dx \frac{1-x^n}{1-x} - \log x \int x^{\frac{b}{a}} dx \log x \frac{1-x^n}{1-x} + \frac{1}{2}\int x^{\frac{b}{a}} dx \frac{1-x^n}{1-x}(\log x)^2}{a^3 x^{\frac{b}{a}}};$$

But if the general term was

$$\frac{x^n}{(an+b)^4},$$

it will be

$$s = \frac{\left\{\begin{array}{c}(\log x)^3 \int x^{\frac{b}{a}} dx \dfrac{1-x^n}{1-x} - 3(\log x)^2 \int x^{\frac{b}{a}} dx \log x \dfrac{1-x^n}{1-x} \\ +3\log x \int x^{\frac{b}{a}} dx (\log x)^2 \dfrac{1-x^n}{1-x} - \int x^{\frac{b}{a}} dx (\log x)^3 \dfrac{1-x^n}{1-x}\end{array}\right\}}{6a^4 x^{\frac{b}{a}}}.$$

From this it is clear, how the value of s looks for the remaining powers ; for, generally, if the general term is

$$\frac{x^n}{(an+b)^m},$$

the sum will be

$$\frac{\left\{\begin{array}{c}(\log x)^{m-1} \int x^{\frac{b}{a}} dx \dfrac{1-x^n}{1-x} - \dfrac{m-1}{1}(\log x)^{m-2} \int x^{\frac{b}{a}} dx \log x \dfrac{1-x^n}{1-x} \\ +\dfrac{m-1}{1} \cdot \dfrac{m-2}{2}(\log x)^{m-3} \int x^{\frac{b}{a}} dx (\log x)^2 \dfrac{1-x^n}{1-x} - \text{etc.}\end{array}\right\}}{1 \cdot 2 \cdot 3 \cdots (m-1) a^m x^{\frac{b}{a}}}.$$

These values become a lot simpler if one puts $x = 1$; for, it will be $\log x = 0$. For, to the general term $\frac{1}{(an+b)^2}$ this summatory term corresponds

$$\frac{\int x^{\frac{b}{a}} dx \log \frac{1}{x} \frac{1-x^n}{1-x}}{1a^2},$$

to the general term $\frac{1}{(an+b)^3}$ this

$$\frac{\int x^{\frac{b}{a}} dx \left(\log \frac{1}{x}\right)^2 \frac{1-x^n}{1-x}}{1 \cdot 2a^3}$$

and to the general term $\frac{1}{(an+b)^m}$ this

$$\frac{\int x^{\frac{b}{a}} dx \left(\log \frac{1}{x}\right)^{m-1} \frac{1-x^m}{1-x}}{1 \cdot 2 \cdot 3 \cdots (m-1) a^m} = \frac{\int x^{\frac{b}{a}} dx \left(\log \frac{1}{x}\right) \frac{1-x^n}{1-x}}{a^m \int dx \left(\log \frac{1}{x}\right)^{m-1}},$$

which integrals must be taken in such a way that for $x = 0$ the whole sum vanishes; but then one must set $x = 1$ and the resulting quantity will be the true sum. Further, note, if the sum of the progression continued to infinity is desired, that one just has to write $\frac{1}{1-x}$ instead of $\frac{1-x^n}{1-x}$ everywhere.

§**13** Two classes of progressions have already been treated, the first of which had the general term Ax^n, the other $\frac{x^n}{A}$, where A denotes an algebraic quantity consisting of n and constants, nevertheless in such a way that n has only positive integer exponents. From this a third class originates having the general term $\frac{Ax^n}{B}$, where A and B denote algebraic quantities of the same kind. Such a progression is reduced to the geometric progression by cancelling the numerator A by means of integration and the denominator B by means of differentiation, as it was done in each of them treated separately. Let the general term of the progression that is to be summed be

$$\frac{(\alpha n + \beta)x^n}{an + b},$$

put the summatory term s; it will be

$$s = \frac{(\alpha + \beta)x}{a + b} + \cdots + \frac{(\alpha n + \beta)x^n}{an + b}.$$

Multiply this equation by px^π; it will be

$$px^\pi s = \frac{p(\alpha + \beta)x^{\pi+1}}{a + b} + \cdots + \frac{p(\alpha n + \beta)x^{n+\pi}}{an + b};$$

take the differentials; it will be

$$pd(x^\pi s) = \frac{p(\pi + 1)(\alpha + \beta)x^\pi dx}{a + b} + \cdots + \frac{p(n + \pi)(\alpha n + \beta)x^{n+\pi-1}dx}{an + b};$$

let

$$pn + p\pi = an + b;$$

it will be

$$p = a \quad \text{and} \quad \pi = \frac{b}{a}.$$

Therefore,

$$ad\left(x^{\frac{b}{a}}s\right) = (\alpha+\beta)x^{\frac{b}{a}}dx + \cdots + (\alpha n+\beta)x^{\frac{b}{a}+n-1}dx.$$

Multiply by px^{π} again; it will be

$$apx^{\pi}d\left(x^{\frac{b}{a}}s\right) = p(\alpha+\beta)x^{\frac{b}{a}+\pi}dx + \cdots + p(\alpha n+\beta)x^{\frac{b}{a}+\pi+n-1}dx.$$

Take integrals; one will have

$$ap\int x^{\pi}d\left(x^{\frac{b}{a}}s\right) = \frac{ap(\alpha+\beta)x^{\frac{b}{a}+\pi+1}}{b+a\pi+a} + \cdots + \frac{ap(\alpha n+\beta)x^{\frac{b}{a}+\pi+n}}{b+a\pi+an}.$$

Let

$$a\alpha pn + a\beta b = an + a\pi + b;$$

it will be

$$p = \frac{1}{\alpha} \quad \text{and} \quad \pi = \frac{\beta}{\alpha} - \frac{b}{a}.$$

Therefore, since

$$\frac{a}{\alpha}\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d\left(x^{\frac{b}{a}}s\right) = x^{\frac{\beta}{\alpha}+1} + \cdots + x^{\frac{\beta}{\alpha}+\pi} = x^{\frac{\beta}{\alpha}+1}\frac{1-x^n}{1-x}.$$

From this equation it results

$$s = \frac{\alpha\int x^{\frac{b}{a}-\frac{\beta}{\alpha}}d\left(x^{\frac{\beta+\alpha}{\alpha}}\frac{1-x^n}{1-x}\right)}{ax^{\frac{b}{a}}}.$$

If the general term was

$$\frac{(\alpha n+\beta)(\gamma n+\delta)x^n}{an+b}$$

and its summatory term is put s, having done the same operations it will result

$$\frac{a}{\alpha}\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d\left(x^{\frac{b}{a}}s\right) = (\gamma+\delta)x^{\frac{\beta}{\alpha}+1} + \cdots + (\gamma n+\delta)x^{\frac{\beta}{\alpha}+n};$$

multiply by px^{π} again and take integrals; it will result

$$\frac{ap}{\alpha}\int x^{\pi}dx\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d\left(x^{\frac{b}{a}}s\right)=\frac{\alpha p(\gamma+\delta)x^{\frac{\beta}{\alpha}+\pi+2}}{\beta+\alpha\pi+2\alpha}+\cdots+\frac{\alpha p(\gamma n+\delta)x^{\frac{\beta}{\alpha}+\pi+n+1}}{\beta+\alpha\pi+\alpha n+\alpha}.$$

Let

$$\alpha\gamma pn+\alpha\delta p=\alpha+\beta+\alpha\pi+\alpha n;$$

it will be

$$p=\frac{1}{\gamma}\quad\text{and}\quad\pi=\frac{\delta}{\gamma}-\frac{\beta}{\alpha}-1.$$

Therefore,

$$\frac{a}{\alpha\gamma}\int x^{\frac{\delta}{\gamma}-\frac{\beta}{\alpha}-1}dx\int x^{\frac{\beta}{\alpha}-\frac{b}{a}}d\left(x^{\frac{b}{a}}s\right)=x^{\frac{\delta}{\gamma}+1}+\cdots+x^{\frac{\delta}{\gamma}+\pi}=x^{\frac{\delta}{\gamma}+1}\frac{1-x^n}{1-x}.$$

Therefore,

$$s=\frac{\alpha\gamma\int x^{\frac{b}{a}-\frac{\beta}{\alpha}}d\left(x^{\frac{\beta}{\alpha}-\frac{\delta}{\gamma}-1}\right)d\left(x^{\frac{\delta}{\gamma}+1}\frac{1-x^n}{1-x}\right)}{ax^{\frac{b}{a}}dx}.$$

But I do not spend time on the summation of progressions of this kind any longer; for, it suffices to have given the method by which they can all be summed. Nevertheless, what I said in § 11 stills hold, of course, that all progressions of this kind either can be summed algebraically or the sum depends on either real or imaginary logarithms.

§14 Now I proceed to another kind of progressions whose general terms can not be expressed algebraically, but which extend to the class of hypergeometric series, a series of this kind is

$$(\alpha+\beta)x+(\alpha+\beta)(2\alpha+\beta)x^2+\cdots+(\alpha+\beta)(2\alpha+\beta)\cdots(\alpha n+\beta)x^n.$$

Put its sum s and multiply by px^{π}; it will be

$$px^{\pi}s=p(\alpha+\beta)x^{\pi+1}+\cdots+p(\alpha+\beta)(2\alpha+\beta)\cdots(\alpha n+\beta)x^{n+\pi}$$

and the integral of this multiplied by dx

$$p\int x^{\pi}sdx = \frac{p(\alpha+\beta)x^{\pi+2}}{\pi+2} + \cdots + \frac{p(\alpha+\beta)(2\alpha+\beta)(3\alpha+\beta)\cdots(\alpha n+\beta)x^{n+\pi+1}}{n+\pi+1}.$$

Let

$$p\alpha n + p\beta = n+\pi+1;$$

it will be

$$p = \frac{1}{\alpha} \quad \text{and} \quad \pi = \frac{\beta}{\alpha} - 1,$$

whence it results

$$\frac{1}{\alpha}\int x^{\frac{\beta}{\alpha}-1}sdx = x^{\frac{\beta}{\alpha}+1} + (\alpha+\beta)x^{\frac{\beta}{\alpha}+2} + \cdots + (\alpha+\beta)(2\alpha+\beta)\cdots(\alpha(n-1)+\beta)x^{\frac{\beta}{\alpha}+n}.$$

Divide by $x^{\frac{\beta}{\alpha}+1}$; one will have

$$\frac{\int x^{\frac{\beta}{\alpha}-1}sdx}{\alpha x^{\frac{\beta}{\alpha}+1}} - 1 = (\alpha+\beta)x + \cdots + (\alpha+\beta)(2\alpha+\beta)\cdots(\alpha(n-1)+\beta)x^{n-1}.$$

This is the propounded progression truncated by the last term. Therefore, it will be

$$\frac{\int x^{\frac{\beta}{\alpha}-1}sdx}{\alpha x^{\frac{\beta}{\alpha}+1}} - 1 = s - (\alpha+\beta)(2\alpha+\beta)\cdots(\alpha n+\beta)x^{n} = s - Ax^{n}.$$

But I explained forms of this kind in a finite expression in another already read dissertation on general terms of transcendental progressions, from which, if somebody wants to, one can take a finite value for A. Therefore, it will be

$$\int x^{\frac{\beta}{\alpha}-1}sdx = \alpha x^{\frac{\beta}{\alpha}+1} + \alpha x^{\frac{\beta}{\alpha}+1}s - \alpha Ax^{\frac{\beta}{\alpha}+\pi+1}$$

and

$$x^{\frac{\beta}{\alpha}-1}sdx = (\alpha+\beta)x^{\frac{\beta}{\alpha}}dx + (\alpha+\beta)x^{\frac{\beta}{\alpha}}sdx + \alpha x^{\frac{\beta}{\alpha}+1}ds - (\alpha+\beta+\alpha n)Ax^{\frac{\beta}{\alpha}+n}dx$$

or

$$sdx = (\alpha+\beta)xdx + (\alpha+\beta)xsdx + \alpha x^2 dx - (\alpha+\beta+\alpha n)Ax^{n+1}dx.$$

From this equation the found value of s will give the sum of the propounded progression. It can also happen that the factors in the following term are not only increased by one but by two or more. Let always two factors enter additionally that this progression results

$$\begin{gathered}(\alpha+\beta)x + (\alpha+\beta)(2\alpha+\beta)(3\alpha+\beta)x^2 + \cdots \\ +(\alpha+\beta)(2\alpha+\beta)\cdots(\alpha(2n-1)+\beta)x^n.\end{gathered}$$

Call its sum s, it will be

$$p\int x^\pi sdx = \frac{p(\alpha+\beta)x^{\pi+2}}{\pi+2} + \cdots + \frac{p(\alpha+\beta)(2\alpha+\beta)\cdots(\alpha(2n-1)+\beta)x^{n+\pi+1}}{n+\pi+1}.$$

Let

$$2p\alpha n - p\alpha + p\beta = n + \pi + 1;$$

it will be

$$p = \frac{1}{2\alpha} \quad \text{and} \quad \pi = \frac{\beta-3\alpha}{2\alpha}.$$

Hence

$$\frac{\int x^{\frac{\beta-3\alpha}{2\alpha}} sdx}{2\alpha} = x^{\frac{\beta+\alpha}{2\alpha}} + \cdots + (\alpha+\beta)(2\alpha+\beta)\cdots(\alpha(2n-2)+\beta)x^{\frac{n+\beta-\alpha}{2\alpha}}.$$

And again

$$\frac{p\int x^\pi dx \int x^{\frac{\beta-3\alpha}{2\alpha}} sdx}{2\alpha} = \frac{2\alpha p x^{\frac{\beta+3\alpha}{2\alpha}+\pi}}{\beta+3\alpha+2\alpha\pi} + \cdots + \frac{2\alpha p(\alpha+\beta)\cdots(\alpha(2n-2)+\beta)x^{n+\pi+\frac{\beta+\alpha}{2\alpha}}}{\beta+\alpha+2\alpha n+2\alpha\pi}.$$

Let

$$4p\alpha^2 n - 4p\alpha^2 + 2p\alpha\beta = 2\alpha n + 2\alpha\pi + \alpha + \beta;$$

it will be

$$p = \frac{1}{2\alpha} \quad \text{and} \quad \pi = \frac{\beta - 2\alpha}{2\alpha} - \frac{\alpha + \beta}{2\alpha} = -\frac{3}{2},$$

as a logical consequence

$$\frac{\int x^{-\frac{3}{2}} dx \int x^{\frac{\beta - 3\alpha}{2\alpha}} dx}{4\alpha^2 x^{\frac{\beta}{2\alpha}}} - \frac{1}{\beta}$$
$$= (\alpha + \beta)x + \cdots + (\alpha + \beta)(2\alpha + \beta) \cdots (\alpha(2n-3) + \beta)x^{n-1} = s - Ax^n$$

having put

$$A = (\alpha + \beta)(2\alpha + \beta) \cdots (\alpha(2n-1) + \beta).$$

From this equation s becomes known.

§15 In like manner, the operation has to be done, if in the coefficient of the following term three or more new factors enter additionally. About this it is to be noted that always as many integral signs are connected to each other in the resulting equation, as there are factors by which each following term is increased. So, the sum s of the progression

$$(\alpha + \beta)x + \cdots + (\alpha + \beta) \cdots (\alpha(3n-2) + \beta)x^n$$

will be determined from this equation

$$\frac{\int x^{-\frac{4}{3}} dx \int x^{-\frac{4}{3}} dx \int x^{\frac{\beta - 5\alpha}{3\alpha}} s dx}{27\alpha^3 x^{\frac{\beta - \alpha}{3\alpha}}} - \frac{1}{\beta(\beta - \alpha)} = s - (\alpha + \beta) \cdots (\alpha(3n-2) + \beta)x^n.$$

From this, that one can make the induction to the following cases, it is to be noted that $\frac{1}{\beta(\beta - \alpha)}$ is the term before the first of the propounded progression or the one whose index is 0. If the factors, which are multiplied by the power of x, do not constitute an arithmetic progression, but another progression of higher order, the operation must be done similarly; for the sake of an example let the propounded progression be

$$(\alpha + \beta)(\gamma + \delta)x + \cdots + (\alpha + \beta)(2\alpha + \beta) \cdots (\alpha n + \beta)(\gamma + \delta)(2\gamma + \delta) \cdots (\gamma n + \delta)x^n;$$

put its sum s; it will be

$$p\int x^{\pi} s dx = \frac{p(\alpha+\beta)(\gamma+\delta)x^{\pi+2}}{\pi+2} + \cdots + \frac{p(\alpha+\beta)\cdots(\alpha n+\beta)(\gamma+\delta)\cdots(\gamma n+\delta)x^{n+\pi+1}}{n+\pi+1}.$$

Put

$$p\gamma n + p\delta = n + \pi + 1;$$

it will be

$$p = \frac{1}{\gamma} \quad \text{and} \quad \pi = \frac{\delta-\gamma}{\gamma}.$$

Therefore,

$$\frac{\int x^{\frac{\delta-\gamma}{\gamma}} s dx}{\gamma} = (\alpha+\beta)x^{\frac{\delta+\gamma}{\gamma}} + \cdots + (\alpha+\beta)\cdots(\alpha n+\beta)(\gamma+\delta)\cdots(\gamma(n-1)+\delta)x^{n+\frac{\delta}{\gamma}}.$$

Further, it will be

$$\frac{p\int x^{\pi} dx \int x^{\frac{\delta-\gamma}{\gamma}} s dx}{\gamma}$$

$$= \frac{\gamma p(\alpha+\beta)x^{\frac{\delta+2\gamma}{\gamma}+\pi}}{\delta+2\gamma+\pi\gamma} + \cdots + \frac{\gamma p(\alpha+\beta)\cdots(\alpha n+\beta)(\gamma+\delta)\cdots(\gamma(n-1)+\delta)x^{n+\pi+\frac{\delta+\gamma}{\gamma}}}{\gamma n+\pi\gamma+\delta+\gamma}.$$

Let

$$p\alpha\gamma n + p\beta\gamma = \gamma n + \pi\gamma + \delta + \gamma;$$

it will be

$$p = \frac{1}{\alpha} \quad \text{and} \quad \pi = \frac{\beta}{\alpha} - \frac{\delta}{\gamma} - 1 = \frac{\beta\gamma-\alpha\delta-\alpha\gamma}{\alpha\gamma}.$$

Therefore,

$$\frac{\int x^{\frac{\beta\gamma-\alpha\delta-\alpha\gamma}{\alpha\gamma}} dx \int x^{\frac{\delta-\gamma}{\gamma}} dx}{\alpha\gamma}$$

$$= x^{\frac{\beta+\alpha}{\alpha}} + \cdots + (\alpha+\beta)\cdots(\alpha(n-1)+\beta)(\gamma+\delta)\cdots(\gamma(n-1)+\delta)x^{\frac{\beta}{\alpha}+n}.$$

As a logical consequence

$$\frac{\int x^{\frac{\beta\gamma-\alpha\delta-\alpha\gamma}{\alpha\gamma}}dx\int x^{\frac{\delta-\gamma}{\gamma}}sdx}{\alpha\gamma x^{\frac{\beta+\alpha}{\alpha}}}-1=s-ABx^n$$

while

$$A=(\alpha+\beta)\cdots(\alpha n+\beta)\quad\text{and}\quad B=(\gamma+\delta)\cdots(\gamma n+\delta).$$

This is the case, if the general term of the progression which defines the factors is

$$(\alpha n+\beta)(\gamma n+\delta)\quad\text{or}\quad\alpha\gamma n^2+(\alpha\delta+\beta\gamma)n+\beta\delta.$$

Therefore, this form contains all progressions of second order. But the above formula, from which s is defined, is transformed into this one

$$\frac{\int x^{\frac{\delta-\gamma}{\gamma}}sdx}{(\beta\gamma-\alpha\delta)x^{\frac{\delta+\gamma}{\gamma}}}+\frac{\int x^{\frac{\beta-\alpha}{\alpha}}sdx}{(\alpha\delta-\beta\gamma)x^{\frac{\beta+\alpha}{\alpha}}}=1+s-ABx^n.$$

From this the form of the following ones can be understood more easily.

§**16** I will now consider the reciprocals of these series, in which the powers of x are divided by that by which they were multiplied before. Therefore, let the series to be summed be this one

$$\frac{x}{\alpha+\beta}+\frac{x^2}{(\alpha+\beta)(2\alpha+\beta)}+\frac{x^3}{(\alpha+\beta)\cdots(3\alpha+\beta)}+\cdots+\frac{x^n}{(\alpha+\beta)\cdots(\alpha n+\beta)};$$

put is sum s. It will be

$$\frac{pd(x^\pi s)}{dx}=\frac{p(\pi+1)x^\pi}{\alpha+\beta}+\frac{p(\pi+2)x^{\pi+1}}{(\alpha+\beta)(2\alpha+\beta)}+\cdots+\frac{p(\pi+n)x^{\pi+n-1}}{(\alpha+\beta)\cdots(\alpha n+\beta)}.$$

Let

$$pn+p\pi=\alpha n+\beta;$$

it will be

$$p = \alpha \quad \text{and} \quad \pi = \frac{\beta}{\alpha}.$$

Therefore, it will be

$$\frac{\alpha d\left(x^{\frac{\beta}{\alpha}}s\right)}{dx} = x^{\frac{\beta}{\alpha}} + \frac{x^{\frac{\beta}{\alpha}+1}}{\alpha+\beta} + \cdots + \frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)(2\alpha+\beta)\cdots(\alpha(n-1)+\beta)}$$

and therefore

$$\frac{\alpha d\left(x^{\frac{\beta}{\alpha}}s\right)}{x^{\frac{\beta}{\alpha}}dx} = 1 + s - \frac{x^n}{A}$$

having, as before, put

$$A = (\alpha+\beta)\cdots(\alpha n+\beta).$$

This equation expanded will give

$$\alpha x^{\frac{\beta}{\alpha}}ds + \beta x^{\frac{\beta-\alpha}{\alpha}}sdx = x^{\frac{\beta}{\alpha}}dx + x^{\frac{\beta}{\alpha}}sdx - \frac{x^{\frac{\beta}{\alpha}+n}dx}{A},$$

which divided by $x^{\frac{\beta}{\alpha}-1}$ goes over into

$$\alpha x ds + \beta s dx = x dx + x s dx - \frac{x^{n+1}dx}{A}$$

or

$$ds + \frac{\beta s dx}{\alpha x} - \frac{s dx}{\alpha} = \frac{dx}{\alpha} - \frac{x^n dx}{A\alpha}.$$

Multiply this equation by $c^{-\frac{x}{\alpha}}x^{\frac{\beta}{\alpha}}$, where c is the number whose hyperbolic logarithm is 1; it will become integrable and it will result

$$c^{\frac{-x}{\alpha}}x^{\frac{\beta}{\alpha}}s = \frac{1}{\alpha}\int c^{\frac{-x}{\alpha}}x^{\frac{\beta}{\alpha}}dx\left(1 - \frac{x^n}{A}\right)$$

and

$$s = \frac{1}{\alpha}c^{\frac{x}{\alpha}}x^{\frac{-\beta}{\alpha}}\int c^{\frac{-x}{\alpha}}x^{\frac{\beta}{\alpha}}dx\left(1 - \frac{x^n}{A}\right).$$

Therefore, the sum of this progression continued to infinity will be

$$\frac{1}{\alpha}c^{\frac{x}{\alpha}}x^{\frac{-\beta}{\alpha}}\int c^{\frac{-x}{\alpha}}x^{\frac{\beta}{\alpha}}dx = \beta(\beta-\alpha)(\beta-2\alpha)\cdots\alpha c^{\frac{x}{\alpha}}x^{\frac{-\beta}{\alpha}}$$
$$-1+\frac{\beta}{x}-\frac{\beta(\beta-\alpha)}{x^2}-\cdots-\frac{\beta(\beta-\alpha)\cdots\alpha}{x^{\frac{\beta}{\alpha}}}.$$

If $\beta = 0$, the sum will be

$$= c^{\frac{x}{\alpha}} - 1.$$

If $\beta = \alpha$, the sum will be

$$= \frac{\alpha c^{\frac{x}{\alpha}}}{x} - 1 - \frac{\alpha}{x}.$$

But if one puts $\beta = 2\alpha$, the sum of the series will be

$$= \frac{2\alpha^2 c^{\frac{x}{\alpha}}}{x^2} - 1 - \frac{2\alpha}{x} - \frac{2\alpha^2}{x^2}$$

and so forth. From this it is understood that, as often as β is a multiple of α, that the sum of the series can be exhibited by a finite and integrated expression. But if $\frac{\beta}{\alpha}$ becomes a fraction, the found integral formula cannot be assigned.

§17 Let each term increase by two factors; one will have this progression

$$\frac{x}{\alpha+\beta}+\frac{x^2}{(\alpha+\beta)\cdots(3\alpha+\beta)}+\frac{x^3}{(\alpha+\beta)\cdots(5\alpha+\beta)}+\cdots+\frac{x^n}{(\alpha+\beta)\cdots(\alpha(2n-1)+\beta)}.$$

Put its sum s; it will be

$$\frac{pd\,(x^\pi s)}{dx}=\frac{p(\pi+1)x^\pi}{\alpha+\beta}+\frac{p(\pi+2)x^{\pi+1}}{(\alpha+\beta)\cdots(3\alpha+\beta)}+\cdots+\frac{p(\pi+n)x^{\pi+n-1}}{(\alpha+\beta)\cdots(\alpha(2n-1)+\beta)}.$$

Let

$$p\pi + pn = 2\alpha n - \alpha + \beta;$$

it will be

$$p = 2\alpha \quad \text{and} \quad \pi = \frac{\beta - \alpha}{2\alpha},$$

therefore,

$$\frac{2\alpha d\left(x^{\frac{\beta-\alpha}{2\alpha}} s\right)}{dx} = x^{\frac{\beta-\alpha}{2\alpha}} + \frac{x^{\frac{\beta+\alpha}{2\alpha}}}{(\alpha+\beta)(2\alpha+\beta)} + \cdots + \frac{x^{\frac{\beta-3\alpha}{2\alpha}+n}}{(\alpha+\beta)\cdots(\alpha(2n-2)+\beta)}.$$

And again

$$\frac{2\alpha p d\left(x^{\pi} d\left(x^{\frac{\beta-\alpha}{2\alpha}} s\right)\right)}{dx^2}$$

$$= \frac{p(\beta-\alpha+2\alpha\pi)}{2\alpha} x^{\frac{\beta-3\alpha}{2\alpha}+\pi} + \cdots + \frac{p(\beta-3\alpha+2\alpha n+2\alpha\pi)x^{\frac{\beta-5\alpha}{2\alpha}+n+\pi}}{2\alpha(\alpha+\beta)(2\alpha+\beta)\cdots(\alpha(2n-2)+\beta)}.$$

Let

$$p\beta - 3p\alpha + 2p\alpha n + 2p\alpha\pi = 4\alpha^2 n - 4\alpha^2 + 2\alpha\beta;$$

it will be

$$p = 2\alpha \quad \text{and} \quad \pi = \frac{1}{2}.$$

Hence it results

$$\frac{4\alpha^2 d\left(x^{\frac{1}{2}} d\left(x^{\frac{\beta-\alpha}{2\alpha}} s\right)\right)}{dx^2} = \beta x^{\frac{\beta-2\alpha}{2\alpha}} + \cdots + \frac{x^{\frac{\beta-4\alpha}{2\alpha}+n}}{(\alpha+\beta)\cdots(\alpha(2n-3)+\beta)}$$

$$= \beta x^{\frac{\beta-2\alpha}{2\alpha}} + x^{\frac{\beta-2\alpha}{2\alpha}} s - \frac{x^{\frac{\beta-2\alpha}{2\alpha}} + n}{(\alpha+\beta)\cdots(\alpha(2n-1)+\beta)}.$$

In like manner, the operation is to be done, if each term increases by several factors in the denominator. But it is sufficiently clear, if the progression constituting the factors of the denominator, was not an arithmetic one, but algebraic of higher order, how one then has to get to the equation from which the sum is determined. Of course, each factor is to be resolved into simple factors, as it was done in § 15 where the general term of factors is $(\alpha n + \beta)(\gamma n + \delta)$, which contains all equations of second order. But not even this is necessary, if it pleases to operate as follows. For the sake of an example, let this progression be propounded

$$\frac{x}{1}+\frac{x^2}{1\cdot 7}+\frac{x^3}{1\cdot 7\cdot 17}+\frac{x^4}{1\cdot 7\cdot 17\cdot 31}+\cdots+\frac{x^n}{1\cdot 7\cdots(2n^2-1)};$$

put its sum s; it will be

$$\frac{pd\,(x^\pi s)}{dx}=p(\pi+1)x^\pi+\cdots+\frac{p(\pi+n)x^{n+\pi-1}}{1\cdot 7\cdots(2n^2-1)}.$$

And again

$$\frac{pd\,(x^\rho d\,(x^\pi s))}{dx^2}=p(\pi+1)(\pi+\rho)x^{\pi+\rho-1}+\cdots+\frac{p(\pi+n)(n+\pi+\rho-1)x^{n+\pi+\rho-2}}{1\cdot 7\cdots(2n^2-1)}.$$

Let

$$pn^2+2p\pi n+p\rho n-pn+p\pi^2+p\pi\rho-p\pi=2n^2-1;$$

it will be

$$p=2,\quad 4\pi+2\rho-2=0\quad\text{or}\quad\rho=1-2\pi$$

and

$$-2\pi^2=-1\quad\text{or}\quad\pi=\sqrt{\frac{1}{2}}\quad\text{and}\quad\rho=1-\sqrt{2}.$$

Hence one will have

$$\begin{aligned}\frac{2d\left(x^{1-\sqrt{2}}d\left(x^{\frac{1}{2}}s\right)\right)}{dx^2}&=x^{-\sqrt{\frac{1}{2}}}+\cdots+\frac{x^{\frac{2n-2-\sqrt{2}}{2}}}{1\cdot 7\cdots(2n^2-4n+1)}\\&=x^{\frac{-\sqrt{2}}{2}}+x^{\frac{-\sqrt{2}}{2}}\left(s-\frac{x^n}{1\cdot 7\cdots(2n^2-1)}\right).\end{aligned}$$

On the other hand the sum of this series continued to infinity will be found from this equation

$$\begin{aligned}&(2-2\sqrt{2})x^{-\sqrt{2}}d\left(x^{\sqrt{\frac{1}{2}}}s\right)+2x^{1-\sqrt{2}}dd\left(x^{\sqrt{\frac{1}{2}}}\right)\\&=(2-2\sqrt{2})x^{\frac{-\sqrt{2}}{2}}dxds+(\sqrt{2}-2)x^{\frac{-2-\sqrt{2}}{2}}sdx^2+2x^{\frac{2-\sqrt{2}}{2}}dds+2\sqrt{2}x^{\frac{-\sqrt{2}}{2}}dxds\end{aligned}$$

$$+(1-\sqrt{2})x^{\frac{-2-\sqrt{2}}{2}}sdx^2$$

$$= x^{\frac{-\sqrt{2}}{2}}dx^2 + x^{\frac{-\sqrt{2}}{2}}sdx^2 = 2x^{\frac{-\sqrt{2}}{2}}dsdx - x^{\frac{-2-\sqrt{2}}{2}}sdx^2 + 2x^{\frac{2-\sqrt{2}}{2}}dds$$

or

$$2xdds - \frac{sdx^2}{x} + 2dsdx = dx^2 + sdx^2,$$

from which equation all irrationalities vanished.

§18 If the factors of the denominators constitute a progression of powers, I will investigate the sums of the progression this way. For the sake of an example, let

$$\frac{x}{(\alpha+\beta)^2} + \frac{x^2}{(\alpha+\beta)^2(2\alpha+\beta)^2} + \cdots + \frac{x^n}{(\alpha+\beta)^2\cdots(n\alpha+\beta)^2};$$

put the sum s; it will be

$$\frac{pd\left(x^\pi s\right)}{dx} = \frac{p(\pi+1)x^\pi}{(\alpha+\beta)^2} + \cdots + \frac{p(\pi+n)x^{n+\pi-1}}{(\alpha+\beta)^2\cdots(n\alpha+\beta)^2};$$

let

$$p\pi + pn = \alpha n + \beta;$$

it will be

$$p = \alpha \quad \text{and} \quad \pi = \frac{\beta}{\alpha}.$$

Therefore,

$$\frac{\alpha d\left(x^{\frac{\beta}{\alpha}}s\right)}{dx} = \frac{x^{\frac{\beta}{\alpha}}}{\alpha+\beta} + \cdots + \frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)^2\cdots(n\alpha+\beta)}.$$

Further,

$$\frac{\alpha pd\left(x^\pi d\left(x^{\frac{\beta}{\alpha}}s\right)\right)}{dx^2} = \frac{p(\beta+\alpha\pi)x^{\frac{\beta}{\alpha}+\pi-1}}{\alpha(\alpha+\beta)} + \cdots + \frac{p(\beta+\alpha\pi+\alpha n-\alpha)x^{\frac{\beta}{\alpha}+\pi+n-2}}{\alpha(\alpha+\beta)^2\cdots(\alpha n+\beta)}.$$

Let

$$p\alpha n + p\beta + p\alpha\pi - p\alpha = \alpha^2 n + \alpha\beta,$$

therefore,

$$p = \alpha \quad \text{and} \quad \pi = \frac{\alpha}{\alpha} = 1.$$

Hence

$$\begin{aligned}\frac{\alpha^2 d\left(xd\left(x^{\frac{\beta}{\alpha}}s\right)\right)}{dx^2} &= x^{\frac{\beta}{\alpha}} + \cdots + \frac{x^{\frac{\beta}{\alpha}+n-1}}{(\alpha+\beta)^2\cdots(\alpha(n-1)+\beta)} \\ &= x^{\frac{\beta}{\alpha}} + x^{\frac{\beta}{\alpha}}\left(s - \frac{x^n}{(\alpha+\beta)^2\cdots(\alpha n+\beta)^2}\right).\end{aligned}$$

And the sum of the progression continued to infinity will be determined from the equation

$$\frac{\alpha^2 d\left(xd\left(x^{\frac{\beta}{\alpha}}s\right)\right)}{x^{\frac{\beta}{\alpha}}dx^2} = 1 + s.$$

Similarly, if the factors were cubes, the sum of the progression

$$\frac{x}{(\alpha+\beta)^3} + \frac{x^2}{(\alpha+\beta)^3(2\alpha+\beta)^3} + \text{etc. to infinity}$$

s will be found from this equation

$$\frac{\alpha^3 d\left(xd\left(xd\left(x^{\frac{\beta}{\alpha}}s\right)\right)\right)}{x^{\frac{\beta}{\alpha}}dx^3} = 1 + s.$$

And so forth for the following.

§**19** Now let the coefficients of the powers of x be fractions, whose numerators and denominators are products consisting of a certain number of factors increasing for each index. So, let this progression be propounded

$$\frac{a+b}{\alpha+\beta}x + \frac{(a+b)(2a+b)}{(\alpha+\beta)(2\alpha+\beta)}x^2 + \cdots + \frac{(a+b)(2a+b)(3a+b)}{(\alpha+\beta)(2\alpha+\beta)(3\alpha+\beta)}x^n;$$

put its sum s; it will be

$$p\int x^{\pi}sdx = \frac{p(a+b)}{(\pi+2)(\alpha+\beta)}x^{\pi+2} + \cdots + \frac{p(a+b)\cdots(an+b)}{(\pi+n+1)(\alpha+\beta)\cdots(\alpha n+\beta)}x^{\pi+n+1}.$$

Let

$$apn + bp = \pi + n + 1;$$

it will be

$$p = \frac{1}{a} \quad \text{and} \quad \pi = \frac{b-a}{a}$$

and hence

$$\frac{\int x^{\frac{b-a}{a}}sdx}{a} = \frac{x^{\frac{b+a}{a}}}{\alpha+\beta} + \cdots + \frac{(a+b)\cdots(a(n-1)+b)}{(\alpha+\beta)\cdots(\alpha n+\beta)}x^{\frac{b}{a}+n}.$$

And again

$$\frac{pd\left(x^{\pi}\int x^{\frac{b-a}{a}}sdx\right)}{adx}$$
$$= \frac{p(b+a+a\pi)}{a(\alpha+\beta)}x^{\frac{b}{a}+\pi} + \cdots + \frac{p(b+an+a\pi)(a+b)\cdots(a(n-1)+b)}{a(\alpha+\beta)\cdots(\alpha n+\beta)}x^{\frac{b}{a}+\pi+n-1}.$$

Let

$$bp + apn + ap\pi = a\alpha n + a\beta;$$

it will be

$$p = \alpha \quad \text{and} \quad \pi = \frac{\beta}{\alpha} - \frac{b}{a},$$

whence it will be

$$\frac{\alpha d\left(x^{\frac{\beta}{\alpha}-\frac{b}{a}}\int x^{\frac{b-a}{a}}sdx\right)}{adx}$$
$$= x^{\frac{\beta}{\alpha}} + \cdots + \frac{(a+b)\cdots(a(n-1)+b)}{(\alpha+\beta)\cdots(\alpha(n-1)+\beta)}x^{\frac{\beta}{\alpha}+n-1} = x^{\frac{\beta}{\alpha}} + x^{\frac{\beta}{\alpha}}\left(s - \frac{(a+b)\cdots(an+b)}{(\alpha+\beta)\cdots(\alpha n+\beta)}x^{n}\right).$$

From this equation one can determine s. If the sum of the propounded progression continued to infinity is desired is desired, it will be

$$\frac{\alpha d\left(x^{\frac{\beta}{\alpha}-\frac{b}{a}}\int x^{\frac{b-a}{a}}sdx\right)}{adx}=x^{\frac{\beta}{\alpha}}+x^{\frac{\beta}{\alpha}}s$$

$$\frac{\alpha}{a}\left(\frac{\beta}{\alpha}-\frac{b}{a}\right)x^{\frac{\beta}{\alpha}-\frac{b}{a}-1}\int x^{\frac{b-a}{a}}sdx+\frac{\alpha}{a}x^{\frac{\beta}{\alpha}-1}s=x^{\frac{\beta}{\alpha}}+x^{\frac{\beta}{\alpha}}s,$$

which goes over into this one

$$\left(\frac{\beta}{a}-\frac{\alpha b}{aa}\right)x^{-\frac{b}{a}}\int x^{\frac{b-a}{a}}sdx+\frac{\alpha}{a}s=x+xs$$

or into this one

$$\left(\frac{\beta}{a}-\frac{\alpha b}{a^2}\right)\int x^{\frac{b-a}{a}}sdx+\frac{\alpha}{a}x^{\frac{b}{a}}s=x^{\frac{b+a}{a}}+x^{\frac{b+a}{a}}s.$$

This equation differentiated gives

$$\left(\frac{\beta}{b}-\frac{\alpha b}{a^2}\right)x^{\frac{b-a}{a}}sdx+\frac{\alpha}{a}x^{\frac{b}{a}}ds+\frac{\alpha b}{a^2}x^{\frac{b-a}{a}}sdx=\frac{b+a}{a}x^{\frac{b}{a}}dx+x^{\frac{b+a}{a}}ds+\frac{b+a}{a}x^{\frac{b}{a}}sdx,$$

which is reduced to this one

$$\frac{\beta}{a}sdx+\frac{\alpha}{a}xds=\frac{b+a}{a}xdx+x^2ds+\frac{b+a}{a}xsdx$$

or

$$ds+\frac{\beta sdx-(b+a)xsdx}{\alpha x-ax^2}=\frac{(b+a)xdx}{\alpha x-ax^2}.$$

Multiply this equation by

$$c^{\int\frac{\beta dx-(b+a)xdx}{\alpha x+ax^2}}\quad\text{or by}\quad x^{\frac{\beta}{\alpha}}(\alpha-ax)^{\frac{b}{a}-\frac{\beta}{\alpha}+1};$$

it will be

$$x^{\frac{\beta}{\alpha}}(\alpha-ax)^{\frac{b}{a}-\frac{\beta}{\alpha}+1}s=(b+a)\int x^{\frac{\beta}{\alpha}}(\alpha-ax)^{\frac{b}{a}-\frac{\beta}{\alpha}}dx$$

and

$$s=\frac{(b+a)\int x^{\frac{\beta}{\alpha}}(\alpha-ax)^{\frac{b}{a}-\frac{\beta}{\alpha}}dx}{x^{\frac{\beta}{\alpha}}(\alpha-ax)^{\frac{b}{a}-\frac{\beta}{\alpha}+1}}.$$

Therefore, the sum can be assigned algebraically, if either $\frac{\beta}{\alpha}$ or $\frac{b}{a}-\frac{\beta}{\alpha}$ was a positive integer number.

§**20** If the progression was composited from coefficients of x of this kind and algebraic ones, first the algebraic coefficients must be cancelled by differentiation and integration, as it was done there, and the the resulting progression must be treated in the way explained here. For the sake of an example, let this progression be propounded

$$\frac{1x}{1} + \frac{3x^2}{1 \cdot 2} + \frac{5x^3}{1 \cdot 2 \cdot 3} + \cdots + \frac{(2n-1)x^n}{1 \cdot 2 \cdot 3 \cdots n};$$

put its sum s; it will be

$$p \int x^{\pi} s dx = \frac{1 \cdot px^{\pi+2}}{(\pi+2)1} + \cdots + \frac{(2n-1)px^{\pi+n+1}}{(\pi+n+1)1 \cdot 2 \cdot 3 \cdots n}.$$

Let

$$2np - p = \pi + n + 1;$$

it will be

$$p = \frac{1}{2} \quad \text{and} \quad \pi = -\frac{3}{2},$$

from which it will be

$$\frac{\int x^{-\frac{3}{2}} s dx}{2} = \frac{x^{\frac{1}{2}}}{1} + \frac{x^{\frac{3}{2}}}{1 \cdot 2} + \cdots + \frac{x^{n+\frac{1}{2}}}{1 \cdot 2 \cdot 3 \cdots n}.$$

Multiply by $x^{\frac{1}{2}}$; it will be

$$\frac{x^{\frac{1}{2}} \int x^{-\frac{3}{2}} s dx}{2} = \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \frac{x^3}{1 \cdot 2 \cdot 3} + \cdots + \frac{x^n}{1 \cdot 2 \cdot 3 \cdots n},$$

therefore,

$$\begin{aligned} \frac{d\left(x^{\frac{1}{2}} \int x^{-\frac{3}{2}} s dx\right)}{2dx} &= 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2} + \cdots + \frac{x^{n-1}}{1 \cdot 2 \cdot 3 \cdots (n-1)} \\ &= 1 + \frac{x^{\frac{1}{2}} \int x^{-\frac{3}{2}} s dx}{2} - \frac{x^n}{1 \cdot 2 \cdot 3 \cdots n}, \end{aligned}$$

from which equation s will be found. But it will be

$$\frac{\int x^{-\frac{3}{2}} s dx}{4x^{\frac{1}{2}}} + \frac{s}{2x} = 1 + \frac{x^{\frac{1}{2}} \int x^{-\frac{3}{2}} s dx}{2} - \frac{x^n}{1 \cdot 2 \cdot 3 \cdots n}.$$

Put $1 \cdot 2 \cdot 3 \cdots n = A$; further, it will be

$$(1-2x)\int x^{-\frac{3}{2}} s dx = 4x^{\frac{1}{2}} - \frac{2s}{x^{\frac{1}{2}}} - \frac{4x^{n+\frac{1}{2}}}{A}.$$

On the other hand, the sum of the propounded progression if continued to infinity will be defined from this equation

$$\int x^{-\frac{3}{2}} s dx = \frac{4x - 2s}{(1-2x)\sqrt{x}},$$

which differentiated gives

$$\frac{sdx}{x\sqrt{x}} = \frac{2xdx + 4xxdx + sdx - 6sxdx - 2xds + 4x^2 ds}{(1-2x)^2 x\sqrt{x}}$$

or

$$xdx + 2x^2 dx - sxdx - 2sx^2 dx - xds + 2x^2 ds = 0,$$

which is reduced to this one

$$ds + \frac{sdx(1+2x)}{1-2x} = \frac{dx(1+2x)}{1-2x}.$$

This multiplied by $\frac{c^{-x}}{1-2x}$ becomes integrable; but it results

$$\frac{c^{-x}s}{1-2x} = \int \frac{c^{-x}dx(1+2x)}{(1-2x)^2} = \frac{c^{-x}}{1-2x} - 1$$

and hence

$$s = 1 - c^x(1-2x).$$

Hence if $x = \frac{1}{2}$, it will be $s = 1$. And hence

$$1 = \frac{1}{1 \cdot 2} + \frac{3}{1 \cdot 2 \cdot 4} + \frac{5}{1 \cdot 2 \cdot 4 \cdot 8} + \frac{7}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 16} + \text{etc. to inf.}$$

§21 From these explanations it is clear to which progressions which are to be summed the method explained in this dissertation extends: Of course, to all those progressions, which are comprehended by the general term $\frac{AP}{BQ}x^{\alpha n + \beta}$, where A and B denote terms of order n of any algebraic progressions, and P is a product of $\gamma n + \delta$ terms of an arbitrary progression, and likewise Q

is a similar product of $\varepsilon n + \zeta$ terms of another algebraic progression. But in total, the sums of progressions will be found in the three explained ways. For, either a completely algebraic sum results, or a certain quadrature is assigned on which the sum depends. Or thirdly, an equation is found whose variable quantities s and x can not be separated from each other at all, that it is not known whether the progression has an algebraic sum or on the quadrature of which curve it depends. But although this method extends so far, nevertheless innumerable progressions not summable by it can occur, whose sums can certainly either not be assigned by any other method, as this one

$$1 + \frac{1}{3} + \frac{1}{7} + \frac{1}{15} + \cdots + \frac{1}{2^n - 1},$$

or whose sums are even known, as this one

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \text{etc.}$$

while the general term is $\frac{1}{a^\alpha - 1}$, in which a and α denote any integer number except 1, whose sum Goldbach proved to be $= 1$. But since its general terms purposely given this way can not be exhbited, it is not surprising that it can not be summed by this method.